MTH 304 **Homework 1**

Topology **Topics:Topology, Basis, Subbasis, Product topology**

1. Let X be a non-empty set. Check if the following collections of subsets of X are topology on X.
   (a) $\mathcal{T} = \{U \subseteq X : X \setminus U \text{ is finite or all of } X\}$. Further, what happens when the set X is finite?
   (b) $\mathcal{T} = \{U \subseteq X : X \setminus U \text{ is countable or all of } X\}$. Further, what happens when the set X is countable?
   (c) $\mathcal{T} = \{U \subseteq X : X \setminus U \text{ is infinite or emply or all of } X\}$.

2. Let $\mathcal{T}_\alpha,\ \alpha \in \Lambda$, be a family of topologies on X.
   (a) Is the intersection $\cap_\alpha \mathcal{T}_\alpha$ a topology on X?
   (b) Is the union $\cup_\alpha \mathcal{T}_\alpha$ a topology on X?
   (c) Show that there exists a unique smallest topology on X containing all the topologies $\mathcal{T}_\alpha$.
   (d) Show that there exists a unique largest topology on X contained in all the topologies $\mathcal{T}_\alpha$.

3. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{C}$ be a collection of open sets of X such that for each open set $U \in \mathcal{T}$ and every $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Show that $\mathcal{C}$ is a basis for $\mathcal{T}$.

4. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

5. Let $X = \mathbb{R}$ and consider the following collections of sets. Determine which ones form basis and further determine the topologies generated by them.
   (a) $\mathcal{B} = \{(a, b) : a < b, a, b \in \mathbb{Q}\}$
   (b) Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. The collection $\mathcal{B}$ consists of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.
   (c) $\mathcal{B} = \{(-\infty, a) : a \in \mathbb{R}\}$

6. Show that the collection $\mathcal{B} = \{[a, b) : a < b, a, b \in \mathbb{Q}\}$ is a basis that generates a topology different from the lower limit topology.

7. Give three different bases for the standard topology on $\mathbb{R}$.

8. Look at the definition of order topology on an ordered set X. This is given in section 14, page 82-83. Show that the order topology derived from the usual order on $\mathbb{R}$ is the standard topology on $\mathbb{R}$.

9. Consider the set $\mathbb{R} \times \mathbb{R}$ in the dictionary order and describe a basis for the ordered topology on $\mathbb{R} \times \mathbb{R}$.

10. If $\mathcal{B}$ is a basis for the topology on X then the collection

$$\mathcal{B}_Y := \{B \cap Y : B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

11. Let Y be a subspace of X. If U is open in Y and Y in open in X, show that U is open in X.

12. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y?
$A = \{x : 1/2 < |x| < 1\}$
$B = \{x : 1/2 < |x| \leq 1\}$
$C = \{x : 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{N}\}$.

13. Show that the countable collection

$$\{(a, b) \times (c, d) : a < b, C < d, \text{ and} a, b, c, d \in \mathbb{Q}\}$$

is a basis for the standard topology on $\mathbb{R}^2$.

14. Let L be a straight line in the plane $\mathbb{R} \times \mathbb{R}$. Describe the subspace topology on L as a subspace of product topological $\mathbb{R}_l \times \mathbb{R}$ and $\mathbb{R}_l \times \mathbb{R}_l$.